Einstein-Cartan-Sciama-Kibble cosmological models with spinning matter and magnetic field

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Spatially homogeneous cosmological models based on the Einstein-Cartan-Sciama-Kibble theory of spacetime are considered. Exact solutions are obtained representing spinning ideal cosmic fluid in the presence of magnetic fields.

I. INTRODUCTION

The Einstein-Cartan-Sciama-Kibble (ECSK) theory of spacetime,[1-3] in which the spin density of matter becomes the source of spacetime's torsion, has attracted much interest in recent years. From the standpoint of cosmology, this interest stems from the fact that nonsingular cosmological models based on the ECSK theory have been explicitly constructed.[4-8] In most of these models, the spin of the cosmic fluid is assumed to be aligned along a particular direction (see, however, Ref. 9). Since strong primordial magnetic fields are the best candidates for the source of spin alignment,[4] it is of interest to study cosmological models containing both spinning matter and magnetic fields.

In this paper, we consider spatially homogeneous models where the spin of the matter content is aligned along, but not coupled to, the magnetic field present. Since it has been shown[10] that only models of Bianchi types I, II, III, VI_0, and VII_0 of symmetry admit solutions when a source-free magnetic field is present, we restrict our considerations to those models. Exact, though sometimes particular, solutions for a class of such models are obtained. The structure of the paper is the following. In Sec. II an outline of the ECSK theory is given and the basic assumptions incorporated in the construction of the models are explicitly stated. In Sec. III the field equations for types I, II, VI_0, and VII_0 are set up and solved. Section IV contains the field equations for models of Bianchi type III, as well as their solution when $p=\rho$. Each section contains a short discussion of the main features of each of the constructed models.

II. ECSK THEORY

As in general relativity (GR), the model of spacetime in the ECSK theory consists of a four-dimensional manifold, which carries a linear connection and a metric with Lorentz signature. Let $\{e_\alpha\}$, $\alpha=0,1,2,3$, be a set of basis vector fields on this manifold, and $\{\omega^\alpha\}$ the set of one-forms dual to $\{e_\alpha\}$. If $\Gamma^\alpha{}_{\beta\gamma}$ and $g_{\alpha\beta}$ are the coefficients of the linear connection and the metric tensor, respectively, we have

$$\nabla_\alpha e_\beta = \Gamma^\gamma{}_{\beta\alpha} e_\gamma \tag{2.1}$$

and

$$dg_{\alpha\beta} + \omega_{\alpha\beta} + \omega_{\beta\alpha} = 0\,, \tag{2.2}$$

where ∇ and d denote the covariant and exterior derivative operations, respectively, while

$$\omega_{\alpha\beta} \equiv g_{\alpha\mu}\omega^\mu{}_\beta\,, \quad \omega^\alpha{}_\beta \equiv \Gamma^\alpha{}_{\beta\gamma}\omega^\gamma\,. \tag{2.3}$$

On the other hand,

$$d\omega^\alpha = -\tfrac{1}{2} C_{\beta\gamma}{}^\alpha \omega^\beta \wedge \omega^\gamma\,, \tag{2.4}$$

where the coefficients of structure, $C_{\beta\gamma}{}^\alpha$, are obtained from the commutation relations

$$[e_\alpha, e_\beta] = C_{\alpha\beta}{}^\gamma e_\gamma\,. \tag{2.5}$$

The torsion tensor, with components given by

$$T^\alpha{}_{\beta\gamma} \equiv \Gamma^\alpha{}_{\gamma\beta} - \Gamma^\alpha{}_{\beta\gamma} - C_{\beta\gamma}{}^\alpha\,, \tag{2.6}$$

is the geometric feature which distinguishes the ECSK spacetime manifold from that of GR, since the torsion is assumed to vanish identically in the latter theory.

In terms of the quantities already defined we can write Cartan's "equations of structure":

$$T^\alpha \equiv \tfrac{1}{2} T^\alpha{}_{\beta\gamma}\omega^\beta \wedge \omega^\gamma = d\omega^\alpha + \omega^\alpha{}_\beta \wedge \omega^\beta\,, \tag{2.7}$$

$$\Omega^\alpha{}_\beta \equiv \tfrac{1}{2} R^\alpha{}_{\beta\gamma\delta}\omega^\gamma \wedge \omega^\delta = d\omega^\alpha{}_\beta + \omega^\alpha{}_\gamma \wedge \omega^\gamma{}_\beta\,, \tag{2.8}$$

where $R^\alpha{}_{\beta\gamma\delta}$ is the curvature tensor.

The field equations are

$$R^\alpha{}_\beta - \tfrac{1}{2}\delta^\alpha{}_\beta R = t^\alpha{}_\beta \tag{2.9}$$

and

$$T^\alpha{}_{\beta\gamma} - \delta^\alpha_\beta T^\mu{}_{\mu\gamma} - \delta^\alpha_\gamma T^\mu{}_{\beta\mu} = S^\alpha{}_{\beta\gamma}\,, \tag{2.10}$$

where $t^\alpha{}_\beta$ and $S^\alpha{}_{\beta\gamma}$ are the tensors of stress-energy density and spin density, respectively. The units were chosen such that $c=1=8\pi G$.

The solutions of Eqs. (2.9) and (2.10) which are presented in the following sections were obtained under the following assumptions.

(i) Spacetime is spatially homogeneous. The spacelike hypersurfaces of homogeneity, parametrized by the cosmic time variable t, are spanned by a set $\{\sigma^i\}$, $i=1,2,3$, of one-forms, such that

$$d\sigma^i = D_{jk}{}^i \sigma^j \wedge \sigma^k , \qquad (2.11)$$

where the $D_{jk}{}^i$'s are the structure constants of the three-parameter group of isometries which acts simply transitively on these surfaces.

In terms of the invariant set $\{\sigma^i\}$ the metric can be written as

$$ds^2 = -dt \otimes dt + g_{ij}(t)\sigma^i \otimes \sigma^j . \qquad (2.12)$$

(ii) It is further assumed that $g_{ij}(t)$ is of the form

$$g_{ij}(t) = \mathrm{diag}\,(a^2, b^2, c^2) , \qquad (2.13)$$

so that, relative to the frame $\{\omega^\alpha\}$ given by

$$\omega^0 = dt , \quad \omega^1 = a\sigma^1 , \quad \omega^2 = b\sigma^2 , \quad \text{and } \omega^3 = c\omega^3 , \qquad (2.14)$$

the metric becomes

$$ds^2 = \eta_{\alpha\beta}\omega^\alpha \otimes \omega^\beta , \quad \eta_{\alpha\beta} = \mathrm{diag}\,(-1,1,1,1) . \qquad (2.15)$$

(iii) The matter spin is described "classically," i.e.,

$$S^\alpha{}_{\beta\gamma} = u^\alpha \sigma_{\beta\gamma} , \quad \sigma_{\alpha\beta} u^\beta = 0 , \qquad (2.16)$$

where u^α is the cosmic fluid velocity vector, which is orthogonal to the hypersurfaces of homogeneity, so, relative to the frame $\{\omega^\alpha\}$, $u^\alpha = (1,0,0,0)$. Furthermore, the spin field itself is homogeneous, so $\sigma_{\alpha\beta} = \sigma_{\alpha\beta}(t)$ and points along the direction of a magnetic field which is present everywhere in the universe.

(iv) The magnetic field does not couple to the spin of the fluid particles. As shown in Ref. 10 this assumption implies that Maxwell's equations admit a solution only when the group of isometries generating the hypersurfaces of homogeneity is of Bianchi types I, II, III, VI_0, or VII_0. For these groups the invariant basis forms σ^i can be so chosen that $d\sigma^i = 0$, for type I; $d\sigma^1 = \sigma^2 \wedge \sigma^3$, $d\sigma^2 = e\sigma^1 \wedge \sigma^3$, and $d\sigma^3 = 0$ for types II, VI_0, and VII_0, when $e = 0$, 1, and -1, respectively; and $d\sigma^1 = \sigma^1 \wedge \sigma^2$, $d\sigma^2 = 0 = d\sigma^3$ for type III. (See Ref. 11, pp. 110–112 for details. Our relations for types VI_0 and VII_0 follow from those given in Ref. 11 on setting $h=-1$, $\omega^1 + \omega^2 = \sigma^1$, $\omega^1 - \omega^2 = \sigma^2$ and $h = 0$, $\omega^1 = \sigma^2$, $\omega^2 = \sigma^1$ in the expressions given for types VI and VII, respectively.)

In Ref. 10 it was also found that the components h^i of the magnetic field must satisfy the conditions

$$h^i D_{ij}{}^j = 0 , \quad \epsilon^{ijk} D_{jk}{}^l h_l = 0 , \qquad (2.17)$$

where ϵ^{ijk} is the totally antisymmetric symbol. Substitution of the values of the $D_{ij}{}^k$'s given by the above choice of the $d\sigma^i$'s in Eqs. (2.17) yields the following results: When the group is of type II, then $h^1 = 0$, necessarily. Similarly, $h^1 = 0 = h^2$ when the group type is III, VI_0, or VII_0. It follows that we can choose the direction of the magnetic field to be along σ^3 in all cases. Then assumption (iii) and Eq. (2.10) imply that the only nonvanishing components of the torsion tensor are

$$T^0{}_{12} = -T^0{}_{21} = 2s(t) \equiv S^0{}_{12} . \qquad (2.18)$$

On the other hand, the magnetic field two-form $F = h(t)\omega^1 \wedge \omega^2$ must satisfy Maxwell's source-free equations

$$dF = 0 = d{*F} , \qquad (2.19)$$

where $*F$ is the dual of F. From these we obtain

$$h(t) = \sqrt{8\pi} H/ab , \qquad (2.20)$$

where H is a constant. Accordingly, the contribution of the magnetic field to the stress-energy tensor is given by

$$t^F_{\alpha\beta} = \left(\frac{H}{ab}\right)^2 \mathrm{diag}\,(1,1,1,-1) . \qquad (2.21)$$

(v) Finally, the cosmic fluid is taken to be a perfect one, so that, due to (2.15) (see Ref. 8 for a detailed proof) (a) its stress-energy tensor takes the simple form

$$t^M_{\alpha\beta} = \mathrm{diag}\,(\rho, p, p, p) , \qquad (2.22)$$

where $\rho = \rho(t)$ is the energy density measured on the hypersurfaces of homogeneity and $p = p(t)$ the corresponding isotropic pressure, and (b) the conservation equations for spin and matter-energy density become

$$[\ln(sabc)]_{,0} = 0 \qquad (2.23)$$

and

$$\rho_{,0} + (\rho + p)[\ln(abc)]_{,0} = 0 , \qquad (2.24)$$

respectively, where $(\ )_{,0} \equiv d(\ )/dt$. Equation (2.24) reflects the fact that the matter energy of a fluid element is conserved separately from the magnetic field energy. This, in turn, is a consequence of the assumption that the magnetic field does not interact with the dipoles of the cosmic fluid particles.

III. FIELD EQUATIONS FOR BIANCHI TYPES I, II, VI_0, AND VII_0 MODELS

As mentioned in the previous section, when

$$d\sigma^1 = \sigma^2 \wedge \sigma^3 , \qquad d\sigma^2 = e\sigma^1 \wedge \sigma^3 , \qquad (3.1)$$

and

$$d\sigma^3 = 0 ,$$

we have a group of Bianchi types II, VI_0, and VII_0 for $e = 0$, 1, and -1, respectively.

Combining (3.1) with (2.12) we obtain

$$
\begin{aligned}
&d\omega^0 = 0\,,\\
&d\omega^1 = (\ln a)_{,0}\omega^0\wedge\omega^1 + \frac{a}{bc}\omega^2\wedge\omega^3\,,\\
&d\omega^2 = (\ln b)_{,0}\omega^0\wedge\omega^2 + \frac{eb}{ac}\omega^1\wedge\omega^3\,,\\
&d\omega^3 = (\ln c)_{,0}\omega^0\wedge\omega^3\,.
\end{aligned}
\tag{3.2}
$$

Using (3.2), (2.7), (2.18) and the fact that $\omega_{\alpha\beta} = -\omega_{\beta\alpha}$, which follows from (2.15) and (2.2), we obtain

$$
\begin{aligned}
&\omega_{10} = (\ln a)_{,0}\omega^1 - s\omega^2\,,\\
&\omega_{20} = s\omega^1 + (\ln b)_{,0}\omega^2\,,\\
&\omega_{30} = (\ln c)_{,0}\omega^3\,,\\
&\omega_{21} = s\omega^0 + \left(\frac{eb}{2ac} - \frac{a}{2bc}\right)\omega^3\,,\\
&\omega_{31} = \left(\frac{a}{2bc} + \frac{eb}{2ac}\right)\omega^2\,,\\
&\omega_{23} = -\left(\frac{a}{2bc} + \frac{eb}{2ac}\right)\omega^1\,.
\end{aligned}
\tag{3.3}
$$

The curvature two-forms are determined from the above expressions for the $\omega_{\alpha\beta}$'s, by using the equations of structure (2.7) and (2.8). The final result, with (2.21) and (2.22) taken into account, is (see Ref. 8 for details)

$$
\begin{aligned}
-R_{00} &= [\ln(abc)]_{,00} + [(\ln a)_{,0}]^2 + [(\ln b)_{,0}]^2 + [(\ln c)_{,0}]^2 - 2s^2\\
&= -\tfrac{1}{2}(\rho + 3p) - \frac{H^2}{a^2b^2}\,,\\
R_{11} &= (\ln a)_{,00} + (\ln a)_{,0}[\ln(abc)]_{,0} + \frac{a^4 - e^2b^4}{2a^2b^2c^2}\\
&= \tfrac{1}{2}(\rho - p) + \frac{H^2}{a^2b^2}\,,\\
R_{22} &= (\ln b)_{,00} + (\ln b)_{,0}[\ln(abc)]_{,0} + \frac{e^2b^4 - a^4}{2a^2b^2c^2}\\
&= \tfrac{1}{2}(\rho - p) + \frac{H^2}{a^2b^2}\,,\\
R_{33} &= (\ln c)_{,00} + (\ln c)_{,0}[\ln(abc)]_{,0} - \frac{(a^2 + eb^2)^2}{2a^2b^2c^2}\\
&= \tfrac{1}{2}(\rho - p) - \frac{H^2}{a^2b^2}\,,\\
R_{12} &= -s[\ln(sb^2c)]_{,0} = 0 = s[\ln(sa^2c)]_{,0} = R_{21}\,.
\end{aligned}
\tag{3.4}
$$

The last two parts of Eqs. (3.4) imply that

$$
s = \frac{s_0}{a^2c}\,,\quad a = Ab\,,
\tag{3.5}
$$

where s_0 and A are constants. Comparing, on the other hand, the R_{11} and R_{22} equations of (3.4) and using (3.5), we obtain

$$
a^4 - e^2b^4 = 0\,,
\tag{3.6}
$$

whereby we conclude that no solution is possible for the Bianchi type-II model ($e = 0$), while $a = b$ for types VI_0 and VII_0. This implies that the models of the latter types are axially symmetric about the spin-magnetic field axis.

On the basis of the above results Eqs. (3.4) reduce to

$$
\begin{aligned}
&(\ln a)_{,00} + (\ln a)_{,0}[\ln(a^2c)]_{,0} = \tfrac{1}{2}(\rho - p) + \frac{H^2}{a^4}\,,\\
&(\ln c)_{,00} + (\ln c)_{,0}[\ln(a^2c)]_{,0} - \frac{k^2}{2c^2} = \frac{1}{2}(\rho - p) - \frac{H^2}{a^4}\,,\\
&(\ln a)_{,0}[\ln(ac^2)]_{,0} - \frac{k^2}{4c^2} + \frac{s_0{}^2}{a^4c^2} = \rho + \frac{H^2}{a^4}\,,
\end{aligned}
\tag{3.7}
$$

where $k = 1 + e$.

A. Type VII_0-I models

The VII_0 model, corresponding to $k = 0$, also admits a group of motions of type I, since it is axially symmetric.[8,12] Thus, we are led to Raychaudhuri's magnetic universes with torsion.[13]

In the case of dust, $p = 0$, it follows from (2.24) that

$$
\rho = \frac{M}{R^3}\,,\quad R^3 \equiv a^2c\,,
\tag{3.8}
$$

where M is a constant, and Eqs. (3.7) can be reduced to the following equation for the scale factor R:

$$
6\frac{R_{,00}}{R} + 30\left(\frac{R_{,0}}{R}\right)^2 - 24\frac{MtR_{,0}}{R^4} - \frac{M}{R^3} + \frac{6M^2t^2 - 2s_0{}^2}{R^6} = 0\,.
\tag{3.9}
$$

Similarly, in the case of "stiff matter," we have

$$
p = \rho = \frac{M}{R^6}\,,
\tag{3.10}
$$

and the equation for R becomes

$$
6\frac{R_{,00}}{R} + 30\left(\frac{R_{,0}}{R}\right)^2 - \frac{24\alpha R_{,0}}{R^4} + \frac{6\alpha^2 + 2M - 2s_0{}^2}{R^6} = 0\,,
\tag{3.11}
$$

where α is a constant of integration. Raychaudhuri[12] obtained for (3.9) only a particular solution, which has the form

$$
R^3 = (s_0{}^2t^2 + \mu^2)^{1/2}\,,
\tag{3.12}
$$

and is valid in the neighborhood of $t = 0$, where R^3 obtains the minimum value $R_{\min}{}^3 = \mu < 1\ \text{cm}^3$. He also gives the general solution of (3.11), where

R^3 obtains a minimum only if $s_0^2 > M + 3\alpha^2$, this being the case when spin dominates over the effects of gravity and shear.

B. Type-VI_0 models

The nonvanishing of the terms of Eqs. (3.7) involving the constant k in the case of the type-VI_0 models makes the above equations quite complicated. A simplification can be obtained with the assumption $p=\gamma\rho$, with γ a constant, and the use of the new time variable τ, defined by $d\tau=(a^2/c)\,dt$. From the conservation equation (2.24) we obtain

$$\rho=\frac{M}{(a^2c)^{1+\gamma}}, \tag{3.13}$$

where M is a constant. Adding and subtracting the first two of Eqs. (3.7), we obtain, respectively,

$$\{a^4[\ln(ac)]'\}'=\frac{(1-\gamma)M}{a^{2(1+\gamma)}c^{\gamma-1}}+\frac{k^2}{2} \tag{3.14}$$

and

$$\{a^4[\ln(a/c)]'\}'=\frac{2H^2c^2}{a^4}-\frac{k^2}{2}, \tag{3.15}$$

where the prime denotes differentiation with respect to τ.

Finally, the last of Eqs. (3.7) becomes

$$(\ln a)'[\ln(ac^2)]'=\frac{k^2}{4a^4}+\frac{M}{a^{2(3+\gamma)}c^{\gamma-1}}+\frac{H^2c^2}{a^8}-\frac{s_0^2}{a^8}. \tag{3.16}$$

We first consider the solution of Eqs. (3.14)–(3.16) for the case of ultrarelativistic fluid, when $\gamma=1$. Equation (3.14) gives

$$[\ln(ac)]'=\frac{k^2(\tau-\tau_0)}{2a^4}, \tag{3.17}$$

where τ_0 is an integration constant which we set equal to zero for convenience. Substituting (3.16) and (3.17) into (3.15), we obtain, after some algebra,

$$\frac{a''}{a}+\frac{(a')^2}{a^2}-\frac{k^2\tau a'}{a^5}+\frac{k^2}{4a^4}-\frac{s_0^2-M}{a^8}=0. \tag{3.18}$$

Although we were not able to arrive at the general solution of (3.18), one easily verifies that the expression

$$a^4=\alpha\tau^2+\beta, \tag{3.19}$$

where α and β are constants, satisfies the above equation, provided

$$\alpha=\frac{k^2}{3},\quad \beta=\frac{12(s_0^2-M)}{5k^2}. \tag{3.20}$$

It then follows from Eq. (3.17) that

$$c=la^2, \tag{3.21}$$

where l is a constant, such that $3(lH)^2=2$, according to (3.16).

For the case of dust, $\gamma=0$, we also obtained only a particular solution. Assuming that (3.21) holds in this case, too, it follows from (3.14) that

$$a^4=\frac{k^2+2lM}{3}(\tau-\tau_0)^2+\epsilon, \tag{3.22}$$

where τ_0 and ϵ are integration constants. Equation (3.15), on the other hand, gives the condition

$$6H^2l^2+Ml-k^2=0, \tag{3.23}$$

while (3.16) determines the value of ϵ:

$$\epsilon=\frac{4s_0^2}{4H^2l^2+4Ml+k^2}. \tag{3.24}$$

Choosing $l=1$, for convenience, we can summarize the above particular solutions of Eqs. (3.14)–(3.16) in the concise form

$$c^2=a^4=\begin{cases}2H^2\tau^2+\dfrac{2(s_0^2-M)}{5H^2} & \text{for } \gamma=1\,,\\[2ex] (M+2H^2)\tau^2+\dfrac{4s_0^2}{5(M+2H^2)} & \text{for } \gamma=0\,.\end{cases} \tag{3.25}$$

The solution for dust ($\gamma=0$) reduces to the one obtained in Ref. 8 when no magnetic field is present ($H=0$). Equation (3.25) for $\gamma=0$ shows that the effect of the magnetic field is to reduce the value of the minimum volume. In the case of ultrarelativistic matter ($\gamma=1$) the presence of the magnetic field is essential in producing the possibility of a nonsingular solution, since the general solution of the field equations when $H=0$ is singular.[8] It follows from (3.25) that it is also necessary that $s_0^2>M$ for a nonsingular solution to exist, exactly as in Raychaudhuri's Bianchi type-I model quoted in Sec. III A.

IV. THE CASE OF BIANCHI TYPE-III MODELS

For Bianchi type-III models the exterior derivatives of the invariant forms σ^i are

$$d\sigma^1=\sigma^1\wedge\sigma^2,\quad d\sigma^2=0=d\sigma^3. \tag{4.1}$$

Then,

$$\begin{aligned} d\omega^1&=(\ln a)_{,0}\,\omega^0\wedge\omega^1+b^{-1}\omega^1\wedge\omega^2,\\ d\omega^2&=(\ln b)_{,0}\,\omega^0\wedge\omega^2,\\ &\text{and}\\ d\omega^3&=(\ln c)_{,0}\,\omega^0\wedge\omega^3. \end{aligned} \tag{4.2}$$

Using the first set, (2.7), of the equations of structure, we find that

$$\begin{aligned}
\omega_{10} &= (\ln a)_{,0}\omega^1 - s\omega^2 , \\
\omega_{20} &= (\ln b)_{,0}\omega^2 + s\omega^1 , \\
\omega_{30} &= (\ln c)_{,0}\omega^3 , \\
\omega_{21} &= s\omega^0 + b^{-1}\omega^1 , \\
\omega_{13} &= 0 = \omega_{23} .
\end{aligned} \tag{4.3}$$

Substitution of expressions (4.3) into the second set, (2.8), of the structure equations yields, after some algebra, the curvature two-forms and the Ricci tensor. As a final result we obtain the field equations in the following form:

$$\begin{aligned}
-R_{00} &= [\ln(abc)]_{,00} + [(\ln a)_{,0}]^2 + [(\ln b)_{,0}]^2 + [(\ln c)_{,0}]^2 - 2s^2 \\
&= -\tfrac{1}{2}(\rho + 3p) - \frac{H^2}{a^2b^2} , \\
R_{02} &= b^{-1}[\ln(ab^{-1})]_{,0} = R_{20} = 0 , \\
R_{11} &= (\ln a)_{,00} + (\ln a)_{,0}[\ln(abc)]_{,0} - \frac{1}{b^2} \\
&= \tfrac{1}{2}(\rho - p) + \frac{H^2}{a^2b^2} , \\
R_{12} &= -s[\ln(sb^2c)]_{,0} = 0 = R_{21} = s[\ln(sa^2c)]_{,0} , \\
R_{22} &= (\ln b)_{,00} + (\ln b)_{,0}[\ln(abc)]_{,0} - \frac{1}{b^2} \\
&= (\rho - p) + \frac{H^2}{a^2b^2} , \\
R_{33} &= (\ln c)_{,00} + (\ln c)_{,0}[\ln(abc)]_{,0} \\
&= \tfrac{1}{2}(\rho - p) - \frac{H^2}{a^2b^2} .
\end{aligned} \tag{4.4}$$

It follows that

$$s = s_0/a^2c , \quad a = Ab , \tag{4.5}$$

where s_0 and A are constants. We set $A = 1$, which is equivalent, according to (4.1), to using $(\sigma^1)' \equiv A\sigma^1$ in place of σ^1 in our calculations.

Thus, Eqs. (4.4) reduce to

$$\begin{aligned}
&(\ln a)_{,00} + (\ln a)_{,0}[\ln(a^2c)]_{,0} - \frac{1}{a^2} = \frac{1}{2}(\rho - p) + \frac{H^2}{a^4} , \\
&(\ln c)_{,00} + (\ln c)_{,0}[\ln(a^2c)]_{,0} = \frac{1}{2}(\rho - p) - \frac{H^4}{a^4} , \\
&(\ln a)_{,0}[\ln(ac^2)]_{,0} = \frac{1}{a^2} + \frac{H^2}{a^4} + \rho - \frac{s_0^2}{a^4c^2} .
\end{aligned} \tag{4.6}$$

I was able to obtain the general solution of Eqs. (4.6) only for the case of stiff matter, when $p = \rho$. Changing variables, according to

$$x = ac , \quad dt = a^2cd\tau \tag{4.7}$$

brings Eqs. (4.6) to the form

$$\begin{aligned}
&(\ln x)'' = x^2 , \\
&(\ln c)'' = -H^2c^2 , \\
&[(\ln x)']^2 - [(\ln c)']^2 = x^2 + H^2c^2 + M - s_0^2 ,
\end{aligned} \tag{4.8}$$

where a prime denotes derivative with respect to τ.

The first two of Eqs. (4.8) have as first integrals

$$\begin{aligned}
&[(\ln x)']^2 = x^2 + C_1 , \\
&[(\ln c)']^2 = -H^2c^2 + C_2 ,
\end{aligned} \tag{4.9}$$

where C_1 and C_2 are integration constants. The first-order equation in (4.8) gives the condition

$$C_1 = C_2 + M - s_0^2 . \tag{4.10}$$

Since c is a real function of τ, C_2 must be positive. Setting $C_2 = l^2$, we solve the second of Eqs. (4.9) and obtain

$$c = \frac{l}{H}\operatorname{sech}[l(\tau - \tau_0)] , \tag{4.11}$$

where τ_0 is a constant of integration.

The solution of the first of Eqs. (4.9) depends on C_1 being less than, equal to, or greater than zero. Respectively, we obtain

$$\begin{aligned}
&x = k\sec(k\tau) \quad \text{for } C_1 = -k^2 , \\
&x = |\tau|^{-1} \quad \text{for } C_1 = 0 , \\
&x = k\operatorname{csch}(k|\tau|) \quad \text{for } C_1 = k^2 ,
\end{aligned} \tag{4.12}$$

where the integration constant was set equal to zero, as it only determines the point on the τ axis where the argument of the above function vanishes.

It follows from Eqs. (4.7), (4.11), and (4.12) that when $C_1 \geq 0$ the volume of this model universe vanishes for some finite value of t. When $C_1 < 0$, however, the model is nonsingular. It starts contracting at $t = -\infty$, goes through a density maximum, and then expands again as $t \to +\infty$.

When no magnetic field is present, then it follows from Eq. (4.9) that the expression (4.11) for c is replaced by

$$c = c_0 e^{\pm l\tau} , \tag{4.13}$$

where c_0 is constant, while the expressions (4.12) for x are still valid. Comparison of this solution with the previous one, where a magnetic field is present, illustrates the fact that such a field acts as a tension along its own direction.[14,15] This is the physical reason why the presence of the magnetic field leads to the reversal of expansion along its direction, illustrated by going from expression (4.13) to (4.11).

[1]A. Trautman, Ann. N. Y. Acad. Sci. 262, 241 (1975).
[2]F. W. Hehl, Gen. Relativ. Gravit. 4, 333 (1974).
[3]F. W. Hehl, Gen. Relativ. Gravit. 5, 491 (1974).
[4]A. Trautman, Nature (London) Phys. Sci. 242, 7 (1973).
[5]W. Kopczynski, Phys. Lett. 43A, 63 (1973).
[6]B. Kuchowicz, Astrophys. Space Sci. 39, 157 (1976).
[7]B. Kuchowicz, Astrophys. Space Sci. 40, 167 (1976).
[8]D. Tsoubelis, Phys. Rev. D 20, 3004 (1979).
[9]B. Kuchowicz, Gen. Relativ. Gravit. 9, 511 (1978).
[10]D. Tsoubelis, Lett. Nuovo Cimento 26, 274 (1979).
[11]M. P. Ryan, Jr. and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, New Jersey, 1975).
[12]G. F. R. Ellis and M. A. H. MacCallum, Commun. Math. Phys. 12, 108 (1969).
[13]A. K. Raychaudhuri, Phys. Rev. D 12, 952 (1975).
[14]D. R. Brill, Phys. Rev. 133, B845 (1964).
[15]D. Tsoubelis, Ph.d. dissertation, The City University of New York, N. Y., 1976 (unpublished).